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Differential equations associated with elliptic surfaces

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Introduction.

The purpose of this paper is to study a certain class of algebraic differential equations which arises in conjunction with the study of elliptic surfaces. The results utilize the general theory of elliptic surfaces due to Kodaira [**11**] and [**12**].

Let E be an elliptic surface over a base curve X. We denote by $\mathcal{J}$ and G the functional and homological invariants of E/X. On a Zariski open subset $X_0 \subset X$, G can be viewed as either a locally constant $\boldsymbol{Z}\oplus\boldsymbol{Z}$ sheaf or as representation $\pi_1(X_0)\rightarrow SL_2(\boldsymbol{Z})$. This representation corresponds to an algebraic vector bundle of rank two on X together with an integrable algebraic connection having regular singular points (Deligne [**2**], Griffiths [**3**]), which is known as the Gauss-Manin connection (Katz and Oda [**10**]). It can also be expressed as a second order algebraic differential equation on X having regular singular points. The aim of this research is to make explicit which algebraic differential equations on X arise from elliptic surfaces in this manner and to investigate certain geometric and arithmetic properties of elliptic surfaces by making use of the differential equations with their monodromy representations and conversely. Central to this investigation is the determination of when two elliptic surfaces over X give rise to equivalent homological invariants (representations) and/or when two of the differential equations have equivalent monodromy and therefore give rise to the same flat vector bundle on X.

In Part I we recall some of the notions that we will be dealing with and by direct calculation obtain some information about the differential equations. In Part II we change our viewpoint. Poincaré [**16**] posed the problem of determining the monodromy representation of a given algebraic differential equation on a curve X. That problem remains unsolved. Instead we determine all differential equations with $SL_2(\boldsymbol{Z})$-monodromy and positivity. This is done without reference to elliptic surfaces, but we are able to see in Part III that these equations are precisely those arising from elliptic surfaces. The remainder of

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Part III is devoted to the interplay between the differential equations and the geometry and we are able to answer the questions mentioned above. In addition, we obtain a classification of elliptic surfaces and elliptic surfaces modulo generic isogeny in terms of the differential equations. Also we develop some relationships between the differential equation, with its monodromy representation, and division points on the generic fibre of E/X.

Part I. Generalities

§1. Elliptic surfaces.

We will make use of a number of notions due to Kodaira (Kodaira [**11**] and [**12**]). However, our notation and usage differs in some respects from his; for that reason, this brief section has been included.

Let X denote a proper smooth connected curve over the field of complex numbers $\boldsymbol{C}$ with function field $K(X)$, and let E denote a proper smooth surface/$\boldsymbol{C}$ which is an elliptic surface over X via a projection $\pi: E \to X$. It will always be assumed that E has no exceptional curves of the first kind in the fibres and that E is free of multiple singular fibres.

To any such elliptic surface we can associate two invariants (Kodaira [**11**]). The functional invariant $\mathcal{J}$ is the rational function on X whose value at "good" $x \in X$ is the J-invariant (modular invariant) of the fibre E_x at x. It will be assumed throughout this paper that $\mathcal{J}$ is non-constant. Next, let S be the support of the singular fibres in X and let $X_0 = X - S$. Denote by G the sheaf $R^1\pi_*(\boldsymbol{Z})|X_0$ which is a locally constant $\boldsymbol{Z} \oplus \boldsymbol{Z}$ sheaf on X_0. G has a natural extension to a sheaf on all of X (Kodaira [**11**]) with stalk $\boldsymbol{Z}$ or 0 depending on whether the degeneracy is multiplicative or additive. G on X_0 or X is called the homological invariant. We will also refer to the corresponding representation $\pi_1(X_0) \to SL_2(\boldsymbol{Z})$ as the homological invariant.

An elliptic surface $E \xrightarrow{\pi} X$ will be called a basic elliptic surface if there exists a global section $s: X \to E$ such that $\pi \circ s = 1_X$. This section corresponds to a $K(X)$-rational point on the generic fibre E^{gen}. Thus E^{gen} becomes an elliptic curve over $K(X)$.

§2. Differential equations.

By an algebraic differential equation of second order on a complete smooth connected curve X over $\boldsymbol{C}$ we shall mean an expression

$$\Lambda f = \frac{d^2 f}{dx^2} + P\frac{df}{dx} + Qf = 0 \tag{2.1}$$

where P, Q, x are in the function field of X and $x \notin \boldsymbol{C}$. The function x will be called a parameter; it will furnish us with a local coordinate at all but a finite set of points. Disregarding these points as well as any where P, Q fail to be regular leaves us with a Zariski-open subset X_0 of X. For every $x \in X_0$ the equation has two independent holomorphic solutions f_1, f_2 which form a basis for the space of solutions in a neighborhood of x. If we pick $x_0 \in X_0$ as a base point and a basis f_1, f_2 for the space of solutions at x_0, we can obtain a representation $\pi_1(X_0, x_0) \xrightarrow{\rho_A} GL_2(\boldsymbol{C})$ by analytic continuation. We will denote by

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \longrightarrow \rho_A(\gamma) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

the analytic continuation of the basis f_1, f_2 around $\gamma \in \pi_1(X_0, x_0)$. Note that another choice of basis at x_0 leads to an equivalent representation.

The spaces of local solutions at points in X_0 can be combined to give a locally constant $\boldsymbol{C}^2$ sheaf on X_0 which when tensored with the structure sheaf $\mathcal{O}_{X_0}$ of X_0 leads to a complex analytic vector bundle V_0 over X_0. This bundle admits a complex analytic connection D_0 with the flat sections corresponding to the original locally constant sheaf. The pair V_0, D_0 is referred to as a flat vector bundle. The relation between representations, flat bundles, and differential equations will be assumed familiar to the reader. One can consult Poincaré [**16**], Deligne [**2**], or Griffiths [**3**]. We remark that any flat bundle on X_0 can be given the structure of an algebraic vector bundle with flat algebraic connection having regular singular points. The class of equations that will concern us, namely K-equations, will be shown to have regular singular points.

§ 3. The differential equation associated to an elliptic surface.

Let $E \xrightarrow{\pi} X$ be an elliptic surface (subject to our usual assumptions). After removing the singular fibres we are left with $E_0 \xrightarrow{\pi} X_0$ where π is now a proper smooth morphism. Whenever such a situation occurs (and more generally) there is a natural integrable connection on the relative algebraic DeRham cohomology sheaf of E_0/X_0: the Gauss-Manin connection (Katz and Oda [**10**]). Thus we have

$$D_0 : \mathcal{H}^1_{DR}(E_0/X_0) \longrightarrow \Omega^1_{X_0/C} \otimes_{\mathcal{O}_{X_0/C}} \mathcal{H}^1_{DR}(E_0/X_0) \tag{3.1}$$

where $\mathcal{H}^1_{DR}$ is the first hyperderived functor of the direct image applied to $\Omega^{\cdot}_{E_0/X_0}$, the relative algebraic DeRham complex, that is

$$\mathcal{H}^1_{DR}(E_0/X_0) = \boldsymbol{R}^1\pi_*(\Omega^{\cdot}_{E_0/X_0}) .$$

The stalk of $\mathcal{H}^1_{DR}(E_0/X_0)$ is easily seen to be $\boldsymbol{H}^1(E_x, \Omega^{\cdot}_{E_x/C})$ where $\boldsymbol{H}^{\cdot}$ indicates the hypercohomology of the algebraic DeRham complex $\Omega^{\cdot}_{E_x/C}$. This is known to be canonically isomorphic to the ordinary complex cohomology of the fibre $H^1(E_x, \boldsymbol{C})$ (Katz [9] or Hartshorne [7]). Thus $\mathcal{H}^1_{DR}(E_0/X_0)$ is nothing more than $R^1\pi_*(\boldsymbol{C})$, and we have a flat vector bundle of rank two over X_0.

The purpose of the rest of this section is to calculate an explicit second order algebraic equation which describes this bundle (such an equation is not unique). In this way we will be able to relate certain properties of these differential equations and representations to elliptic surfaces and conversely. We remark that it is enough to work only with basic elliptic surfaces because the flat bundle associated to $E\xrightarrow{\pi}X$ is isomorphic to the one associated to the unique basic surface with the same functional and homological invariant. We therefore restrict our attention to basic surfaces.

Let U be the unit u-disc and suppose we are given a family of elliptic curves E over U described by

$$y^2=4x^3-g_2(u)x-g_3(u) \tag{3.2}$$

where $g_2(u)$, $g_3(u)$ are holomorphic on U with $g_2^3(u)-27g_3^2(u)$ never zero. This family has an obvious global section over U. For $u_0\in U$ the cohomology of the fibre $H^1(E_{u_0}, \boldsymbol{C})$ can be identified with the space of differentials of the first and second kind on E_{u_0} modulo the exact differentials. This space has as basis dx/y and xdx/y. The natural differentiation with respect to u can be extended to this space of differentials by regarding dx/y and xdx/y as dependent on u and setting $D_u(x)=0$. The result is the well-known system of differential equations

$$\frac{d}{du}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}=\begin{pmatrix} \dfrac{-1}{12}\dfrac{\frac{dD}{du}}{D} & \dfrac{3\delta}{2D} \\[2ex] \dfrac{-g_2\delta}{8D} & \dfrac{1}{12}\dfrac{\frac{dD}{du}}{D} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{3.3}$$

where

$$D=g_2^3(u)-27g_3^2(u)$$

and

$$\delta=3g_3(u)\frac{dg_2(u)}{du}-2g_2(u)\frac{dg_3(u)}{du}.$$

Since our family is topologically trivial, we can select some fixed cycle γ on each fibre. One solution of (3.3) then is

$$f_1(u)=\int_\gamma\frac{dx}{y},\qquad f_2(u)=\int_\gamma\frac{xdx}{y}.$$

Suppose now that $E\overset{\pi}{\to}X$ is a basic elliptic surface. Let $X_0=X$ minus the support of the singular fibres and any point where the functional invariant $\mathcal{J}$ takes value 0, 1, ∞ or ord $d\mathcal{J}\neq 0$. Let $x_0\in X_0$ be a base point. As our family is basic, we can locally (near x_0) represent it as

$$y^2=4x^3-g(t)x-g(t) \tag{3.4}$$

where

$$g(t)=\frac{27\mathcal{J}(t)}{\mathcal{J}(t)-1}$$

is holomorphic non-vanishing and further g^3-27g^2 is never zero. Here $t=x-x(x_0)$ where $x\in K(X)$ is a fixed parameter at every point in X_0 (shrink X_0 if necessary).

THEOREM I.3.1. *Near x_0 a differential equation derived from the system* (3.3) *and* (3.4) *is*

$$\Lambda f=\frac{d^2f}{dx^2}+\frac{\left(\frac{d\mathcal{J}}{dx}\right)^2-\mathcal{J}\left(\frac{d^2\mathcal{J}}{dx^2}\right)}{\mathcal{J}\frac{d\mathcal{J}}{dx}}\left(\frac{df}{dx}\right)+\frac{\left(\frac{d\mathcal{J}}{dx}\right)^2\left(\frac{31}{144}\mathcal{J}-\frac{1}{36}\right)}{\mathcal{J}^2(\mathcal{J}-1)^2}f=0 \tag{3.5}$$

where f is unknown and $x\in K(X)$ is a fixed parameter.

PROOF. A suitable holomorphic change of frame in (3.3) is made—followed by a calculation.

Now although this calculation is only local, the result is clearly a global second order algebraic differential equation on X.

THEOREM I.3.2. *This differential equation* (3.5) *has regular singular points.*

PROOF. Just check.

Unfortunately, this global equation may not be the one we want for E/X. This is because the local model (3.4) may not be a global model for E/X (or rather over X_0). We must see how the local models differ on an overlap. There are only two ways: on each fibre either $x\to x$, $y\to y$ the identity, or $x\to x$, $y\to -y$ the involution. This is because the general fibre does not admit complex multiplication and the section must be preserved. We thereby obtain a 1-cocycle on X_0 with $\boldsymbol{Z}_2$ coefficients which gives an element of $H^1(X_0, \boldsymbol{Z}_2)$ or a map $\pi_1(X_0)\to\boldsymbol{Z}_2$. It is easily seen that the differential equation of Theorem I.3.1 yields a representation projectively equivalent (conjugate in $PSL_2(\boldsymbol{Z})$) to that of E/X.

DEFINITION I.3.3. Two elliptic surfaces will be called projectively equivalent if their associated representations are projectively equivalent (on some Zariski open subset of X), that is, conjugate in $PSL_2(\boldsymbol{Z})$.

THEOREM I.3.4. *Every representation associated to an elliptic surface is, up to projective equivalence, given by the monodromy representation of a differential* *equation of the type* (3.5).

§ 4. A simple example.

In $\boldsymbol{P}_C^1 \times \boldsymbol{P}_C^2$ with homogeneous coordinates $(u:v)$ and $(x:y:z)$ respectively, we consider the variety W described by

$$(u-v)y^2z-4x^3(u-v)+27uxz^2+27uz^3=0\,. \tag{4.1}$$

In affine coordinates $(u:1)$, $(x:y:1)$, assuming we avoid $u=0$ or 1, this becomes

$$y^2=4x^3-\frac{27u}{u-1}x-\frac{27u}{u-1}\,. \tag{4.2}$$

Thus $W\to\boldsymbol{P}_C^1$ is a family of elliptic curves over $\boldsymbol{P}_C-\{0, 1, \infty\}$. This family is smooth except over 1 in the u-plane, where the surface has a singularity at $(0:1:0)$. We resolve this to get a smooth surface $E/\boldsymbol{P}_C^1$ which is taken to have no exceptional curves of the first kind in the fibres. Now clearly there is a section over $\boldsymbol{P}_C^1-\{1\}$, and this section extends over all $\boldsymbol{P}_C^1$. Therefore E is basic. The functional invariant is $\mathcal{J}=u$ (on the u-sphere). The differential equation is

$$\frac{d^2f}{du^2}+\frac{1}{u}\frac{df}{du}+\frac{(31/144)u-1/36}{u^2(u-1)^2}f=0 \tag{4.3}$$

which is hypergeometric. Further the singular fibres are:

Type I_1 at ∞ $\quad\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

Type II at 0 $\quad\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}$

Type III* at 1 $\quad\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$

(see Kodaira [**11**] for terminology). The last can be arrived at by explicit resolution. The matrix at the right is the local $SL_2(\boldsymbol{Z})$ matrix around such a singularity. This can be determined from the monodromy representation of the differential equation (4.3), which we compute in Part II, section 2. We will make use of this simple example in a construction in Part III.

Part II. *K*-equations

§ 1. *K*-equations.

Let X be a complete smooth connected algebraic curve over $\boldsymbol{C}$ with function field denoted by $K(X)$. After fixing a parameter $x\in K(X)$, consider an algebraic differential equation on X

$$\Lambda f=\frac{d^2f}{dx^2}+P\frac{df}{dx}+Qf=0 \tag{1.1}$$

with P and Q in $K(X)$ and f an unknown function.

DEFINITION II.1.1. $\Lambda f=0$ is called a K-equation if it possesses two solutions, ω_1 and ω_2, which are holomorphic non-vanishing multivalued functions on some Zariski open subset X_0 of X, satisfying:

(i) ω_1 and ω_2 form a basis of solutions,

(ii) for every closed path $\gamma\in\pi_1(X_0)$ the analytic continuation of $\binom{\omega_1}{\omega_2}$ around γ is $M_\gamma\binom{\omega_1}{\omega_2}$ with $M_\gamma\in SL_2(\boldsymbol{Z})$ (the monodromy representation),

(iii) $\mathrm{Im}\,(\omega_1/\omega_2)>0$ on X_0 (positivity).

Such a pair of solutions is called a K-basis. In addition, since the monodromy is in $SL_2(\boldsymbol{Z})$, the Wronskian $W=e^{-\int Pdx}$ is single-valued. We assume as part of our definition:

(iv) $W\in K(X)$.

Let $\Lambda f=0$ be a K-equation with K-basis ω_1 and ω_2. Consider the function $\mathcal{J}=J\circ\omega_1/\omega_2$,

$$X_0\xrightarrow{\ \omega_1/\omega_2\ }\mathfrak{H}\xrightarrow{\ J\ }\boldsymbol{C}$$

where J is the elliptic modular function on the upper half plane $\mathfrak{H}$. This $\mathcal{J}$ is a single-valued holomorphic function on $X_0\subset X$.

PROPOSITION II.1.2. $\mathcal{J}\in K(X)$.

PROOF. This is an application of a result which appears in Kodaira [**11**] as Theorem 7.3. We remark that the proof shows that where $\mathcal{J}$ has a b^{th} order pole, $b>0$ the local monodromy is conjugate in $SL_2(\boldsymbol{Z})$ to $\pm\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$.

Thus to every K-equation $\Lambda f=0$ and K-basis ω_1, ω_2 we associate a rational function $\mathcal{J}$ in $K(X)$, which is necessarily non-constant. Notice that another basis of solutions $\binom{\tilde{\omega}_1}{\tilde{\omega}_2}=M\binom{\omega_1}{\omega_2}$ with $M\in SL_2(\boldsymbol{Z})$ gives a K-basis yielding the same $\mathcal{J}$ and equivalent monodromy representation, where $\binom{\tilde{\omega}_1}{\tilde{\omega}_2}\to MM_\gamma M^{-1}\binom{\tilde{\omega}_1}{\tilde{\omega}_2}$ under continuation around $\gamma\in\pi_1(X_0)$, with M_γ being the monodromy of $\binom{\omega_1}{\omega_2}$ around γ. Likewise, multiplication by a scalar $\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}$ $a\in\boldsymbol{C}$ changes neither $\mathcal{J}$ nor the representation.

DEFINITION II.1.3. Two K-bases of a fixed K-equation, $\Lambda f=0$, will be called $\boldsymbol{Z}$-equivalent if they differ, as above, by a matrix M which is in $SL_2(\boldsymbol{Z})$, scalar, or a product thereof. This is the same as requiring that they give the same $\mathcal{J}$.

REMARK. If ω_1, ω_2 is a K-basis of the K-equation $\Lambda f=0$, then to each point x in some Zariski open subset X_0 of X we can assign the lattice $L_x=\{m\omega_1(x)+n\omega_2(x)$ with $m, n\in \boldsymbol{Z}\}\subset \boldsymbol{C}$. This lattice is well-defined at $x\in X_0$ independent of the analytic continuation of ω_1, ω_2—continuation giving rise only to automorphisms of L_x. Further, $\boldsymbol{Z}$-equivalence at most adjusts L_x be a constant factor of homothety. We can then construct an analytic family of elliptic curves over X_0 with global section, functional invariant $\mathcal{J}=J(\omega_1/\omega_2)$, and homological invariant G isomorphic to the locally constant $\boldsymbol{Z}\oplus\boldsymbol{Z}$ sheaf associated to the monodromy representation of Λ given by ω_1, ω_2. This family can be uniquely compactified to a basic elliptic surface over X (see Part III and also Neron [**14**] or Kodaira [**11**]). The construction will depend only on Λ and the $\boldsymbol{Z}$-equivalence class of the K-basis selected.

§ 2. Classification of K-equations and regularity.

Let X be any base curve and $\mathcal{J}\in K(X)$ any non-constant rational function on X. The problem is to produce a K-equation $\Lambda f=0$ with a K-basis ω_1 and ω_2 having $\mathcal{J}=J(\omega_1/\omega_2)$.

Consider first the z-sphere $\boldsymbol{P}^1_{\boldsymbol{C}}$ and the hypergeometric differential equation

$$\frac{d^2f}{dz^2}+\frac{1}{z}\frac{df}{dz}+\frac{31/144z-1/36}{z^2(z-1)^2}f=0 \quad . \tag{2.1}$$

The solution in terms of Riemann's P-function is

$$P\begin{Bmatrix} 0 & \infty & 1 \\ -1/6 & 0 & 1/4 \\ 1/6 & 0 & 3/4 \end{Bmatrix}=z^{-1/6}(z-1)^{1/4}P\begin{Bmatrix} 0 & \infty & 1 \\ 0 & 1/12 & 0 \\ 1/3 & 1/12 & 1/2 \end{Bmatrix}$$

which is seen to be a hypergeometric function. Thus at $z=0$ we have two solutions

$$\eta_1=z^{-1/6}(z-1)^{1/4}{}_2F_1\Big(\frac{1}{12}, \frac{1}{12}; \frac{2}{3}; z\Big)$$

$$\eta_2=z^{1/6}(z-1)^{1/4}{}_2F_1\Big(\frac{5}{12}, \frac{5}{12}; \frac{4}{3}; z\Big)$$

which form a basis. We now let

$$c=(2-\sqrt{3})\Big[\frac{\Gamma(11/12)}{\Gamma(7/12)}\Big]^2\frac{\Gamma(2/3)}{\Gamma(4/3)}$$

(Γ the gamma function), and consider another basis of solutions at $z=0$

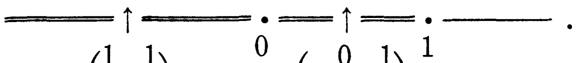
$$\Phi_1 = e^{2\pi i/3}\eta_1 + c\eta_2 \tag{2.2}$$
$$\Phi_2 = \eta_1 - ce^{-\pi i/3}\eta_2 .$$

The quotient of these solution $\Phi(z)=\Phi_1(z)/\Phi_2(z)$ can be regarded as a multivalued function

$$P_C^1 - \{0,\ 1,\ \infty\} \xrightarrow{\ \Phi\ } P_C^1 .$$

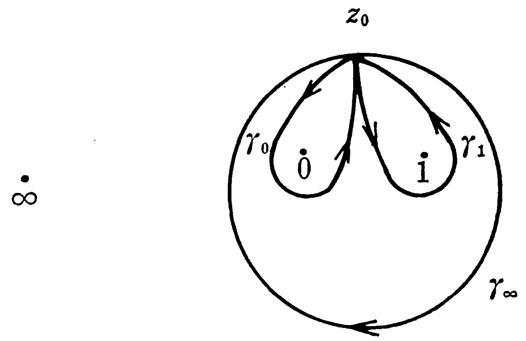
However, Φ is an inverse of the elliptic modular function J (Bateman [1]), that is, $\tau=\Phi(J(\tau))$ for $\tau \in \mathfrak{H}$ the upper half plane. Hence Φ maps $P_C^1-\{0, 1, \infty\}$ to $\mathfrak{H}$ and Φ_1, Φ_2 form a K-basis of solutions for (2.1) which is then a K-equation.

REMARK. The monodromy representation of (2.1) is easily computed. Slit the z-sphere from ∞ to 0 along the negative real axis and then slit from 0 to 1. A single-valued branch of the quotient of solutions on this slit sphere can be selected to take values in the usual fundamental domain $\{\tau \in \mathfrak{H}$ such that $-1/2<\mathrm{Re}\,\tau<1/2$, $|\tau|>1\}$. Continuation across these slits in the directions indicated leads to the monodromy transformations shown (up to sign):



$$\pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \qquad \pm\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

Taking γ_0, γ_1, γ_∞



as generators for $\pi_1(P_C^1-\{0, 1, \infty\}, z_0)$, we have:

$$\gamma_\infty \longrightarrow \pm\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$
$$\gamma_0 \longrightarrow \pm\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix}$$
$$\gamma_1 \longrightarrow \pm\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

However at ∞ the exponents of (2.1) are 0, 0 so the trace is $+2$ and at 0 the exponents are $-1/6$, $1/6$ so the trace is 1. Thus

$$\gamma_\infty \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\gamma_0 \longrightarrow \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\gamma_1 \longrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The last because we must have

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We return to the problem posed at the beginning of this section. X will be our base curve, $\mathcal{J}\in K(X)$ a non-constant rational function, and $x\in K(X)$ a fixed parameter. We regard $\mathcal{J}$ as a map $\mathcal{J}: X\rightarrow \boldsymbol{P}_{\boldsymbol{C}}^1$ onto the sphere. Consider the compositions $\Phi_1(\mathcal{J})$, $\Phi_2(\mathcal{J})$ as multivalued holomorphic non-vanishing functions on some appropriate Zariski open subset X_0 of X. They satisfy a differential equation easily computed to be:

$$\frac{d^2f}{dx^2}+\frac{\left(\frac{d\mathcal{J}}{dx}\right)^2-\mathcal{J}\frac{d^2\mathcal{J}}{dx^2}}{\mathcal{J}\frac{d\mathcal{J}}{dx}}\frac{df}{dx}+\frac{\left(\frac{d\mathcal{J}}{dx}\right)^2\left(\frac{31}{144}\mathcal{J}-\frac{1}{36}\right)}{\mathcal{J}^2(\mathcal{J}-1)^2}f=0 \tag{2.3}$$

which is precisely that of Part I! Clearly

$$\operatorname{Im}\frac{\Phi_1(\mathcal{J})}{\Phi_2(\mathcal{J})}>0, \qquad J\left(\frac{\Phi_1(\mathcal{J})}{\Phi_2(\mathcal{J})}\right)=\mathcal{J}$$

and

$$\begin{pmatrix} \Phi_1(\mathcal{J}) \\ \Phi_2(\mathcal{J}) \end{pmatrix}\longrightarrow M_\gamma\begin{pmatrix} \Phi_1(\mathcal{J}) \\ \Phi_2(\mathcal{J}) \end{pmatrix}$$

$M_\gamma\in SL_2(\boldsymbol{Z})$ under analytic continuation around $\gamma\in\pi_1(X_0)$. Note also that

$$\frac{\left(\frac{d\mathcal{J}}{dx}\right)^2-\mathcal{J}\frac{d^2\mathcal{J}}{dx^2}}{\mathcal{J}\frac{d\mathcal{J}}{dx}}=-\frac{d}{dx}\log\frac{\frac{d\mathcal{J}}{dx}}{\mathcal{J}}$$

so the Wronskian

$$W=e^{-\int\left(-\frac{d}{dx}\log\frac{\frac{d\mathcal{J}}{dx}}{\mathcal{J}}\right)dx}=\frac{\frac{d\mathcal{J}}{dx}}{\mathcal{J}}.$$

As d/dx is a derivation $K(X) \xrightarrow{d/dx} K(X)$ and $\mathcal{J} \in K(X)$ non-constant, we have $W \in K(X)$. Therefore (2.3) is a K-equation and:

THEOREM II.2.1. *Let X be any base curve and $\mathcal{J} \in K(X)$ non-constant. Then there exists a K-equation $\Lambda f=0$ and a K-basis ω_1, ω_2 of its solutions such that $J(\omega_1/\omega_2)=\mathcal{J}$. Those constructed above will be referred to as SK-equations, and denoted by $\Lambda=\Lambda_{(\mathcal{J},1)}$.*

COROLLARY II.2.2. *The monodromy representation of any K-equation with respect to a fixed K-basis of solutions is projectively equivalent (in $PSL_2(\boldsymbol{Z})$) to that of the differential equation (2.3) above with appropriate $\mathcal{J}$.*

PROOF. It is obvious that $\mathcal{J}$ determines the projective monodromy in $PSL_2(\boldsymbol{Z})$ *up to conjugation in $PSL_2(\boldsymbol{Z})$.*

COROLLARY II.2.3. *The monodromy group $\Gamma \subset SL_2(\boldsymbol{Z})$ of a K-equation with respect to a fixed K-basis of solutions has finite index in $SL_2(\boldsymbol{Z})$.*

PROOF. Obvious.

COROLLARY II.2.4. *The homological invariant G of an elliptic surface $E \xrightarrow{\pi} X$ thought of as a map $\pi_1(X_0) \rightarrow SL_2(\boldsymbol{Z})$ has image of finite index in $SL_2(\boldsymbol{Z})$.*

PROOF. Follows from the results in Part I.

We will now determine all K-equations. Fix a K-equation $\Lambda f=0$ on X with K-basis ω_1, ω_2 such that $\mathcal{J}=J(\omega_1/\omega_2)$. Say

$$\Lambda f=\frac{d^2 f}{dx^2}+P\frac{df}{dx}+Qf=0\,. \tag{2.4}$$

We also consider the SK-equation $\tilde{\Lambda}=\tilde{\Lambda}_{(\mathcal{J},1)}$

$$\tilde{\Lambda} f=\frac{d^2 f}{dx^2}+\tilde{P}\frac{df}{dx}+\tilde{Q}f=0 \tag{2.5}$$

with solutions $\Phi_1(\mathcal{J}), \Phi_2(\mathcal{J})$. Let $X_0 \subset X$ be a Zariski open set on which both bases $\Phi_1(\mathcal{J}), \Phi_2(\mathcal{J})$ and ω_1, ω_2 are holomorphic non-vanishing multivalued functions. Select a base point $x_0 \in X_0$. By Corollary II.2.2 the monodromy representations in $SL_2(\boldsymbol{Z})$ of Λ, $\tilde{\Lambda}$ with respect to the chosen K-bases are projectively equivalent, that is, conjugate in $PSL_2(\boldsymbol{Z})$. Altering ω_1, ω_2 to a $\boldsymbol{Z}$-equivalent basis if necessary, we can assume the projective representations $\pi_1(X_0, x_0) \rightarrow PSL_2(\boldsymbol{Z})$ are equal.

THEOREM II.2.5. *There exists an algebraic function λ on X with $\lambda^2 \in K(X)$ such that $\lambda\Phi_1(\mathcal{J})=\omega_1$ and $\lambda\Phi_2(\mathcal{J})=\omega_2$, and therefore $\Lambda f=0$ is determined:*

$$\begin{aligned} P&=\tilde{P}-\frac{d}{dx}\log\lambda^2 \in K(X) \\ Q&=\tilde{Q}-\tilde{P}\frac{d}{dx}\log\lambda-\frac{\frac{d^2\lambda}{dx^2}}{\lambda}+2\left(\frac{\frac{d\lambda}{dx}}{\lambda}\right)^2 \in K(X)\,. \end{aligned} \tag{2.6}$$

PROOF. For every point x near x_0, the lattices $\tilde{L}_x=\boldsymbol{Z}\Phi_1(\mathcal{J}(x))+\boldsymbol{Z}\Phi_2(\mathcal{J}(x))$ and $L_x=\boldsymbol{Z}\omega_1(x)+\boldsymbol{Z}\omega_2(x)$ in $\boldsymbol{C}$ are homothetic. So locally there is a function $\lambda(x)$ holomorphic non-vanishing such that $\lambda(x)\tilde{L}_x=L_x$. Suppose near x_0

$$\begin{aligned}\lambda\Phi_1(\mathcal{J})&=a\omega_1+b\omega_2\\ \lambda\Phi_1(\mathcal{J})&=c\omega_1+d\omega_2\end{aligned} \tag{2.7}$$

with $\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\boldsymbol{Z})$. Now analytically continue around $\gamma\in\pi_1(X_0, x_0)$. If $\tilde{\lambda}$ denotes the continuation of λ, we have:

$$\begin{pmatrix}\tilde{\lambda} & 0\\ 0 & \tilde{\lambda}\end{pmatrix}M_\gamma\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}=\pm\begin{pmatrix}a & b\\ c & d\end{pmatrix}M_\gamma\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$$

where $\pm M_\gamma$ is the monodromy of Λ, $\tilde{\Lambda}$ (assumed projectively equal). Therefore

$$\pm\begin{pmatrix}a & b\\ c & d\end{pmatrix}M_\gamma^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}M_\gamma\begin{pmatrix}\tilde{\lambda} & 0\\ 0 & \tilde{\lambda}\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$$

or by (2.7)

$$\pm\begin{pmatrix}a & b\\ c & d\end{pmatrix}M_\gamma^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}M_\gamma\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}=\begin{pmatrix}\lambda/\tilde{\lambda} & 0\\ 0 & \lambda/\tilde{\lambda}\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}.$$

This implies, as

$$\pm\begin{pmatrix}a & b\\ c & d\end{pmatrix}M_\gamma^{-1}\begin{pmatrix}a & b\\ c & d\end{pmatrix}^{-1}M_\gamma\in SL_2(\boldsymbol{Z}),$$

that $\lambda/\tilde{\lambda}$ is a complex multiplication of $\tilde{L}_x$ for all x near x_0. But since $\mathcal{J}$ is necessarily non-constant, the general fibre does not admit non-trivial complex multiplication. So $\lambda/\tilde{\lambda}$ is identically ±1, and in fact this requires $\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\pm1$. Therefore λ^2 is single-valued on X_0.

One can compute that $\omega_1=\lambda\Phi_1(\mathcal{J})$ and $\omega_2=\lambda\Phi_2(\mathcal{J})$ then satisfy a differential equation with P and Q as in (2.6). Computing the Wronskian gives $e^{-\int Pdx}=e^{-\int\tilde{P}dx}\cdot\lambda^2$ up to a constant multiple. However, $e^{-\int Pdx}$ is the Wronskian W of Λ and $e^{-\int\tilde{P}dx}=\dfrac{d\mathcal{J}/dx}{\mathcal{J}}$. Since we assumed $W\in K(X)$, we have $\lambda^2\in K(X)$.

REMARK. If we drop the assumption that W is rational, we have then classified all differential equations with the properties of $SL_2(\boldsymbol{Z})$ monodromy and positivity, $\mathrm{Im}\,(\omega_1/\omega_2)>0$.

THEOREM II.2.6. *Let $\Lambda f=0$ be a K-equation. Then Λ has regular singular points and essentially unipotent local monodromy.*

PROOF. This may be checked directly from the expression (2.6) for Λ.

Alternatively, since we will show that the equation comes from geometry, we could apply the Monodromy Theorem and the Regularity Theorem (Deligne [2]) to reach our conclusions.

Part III. Elliptic surfaces and K-equations

§ 1. Basic surfaces associated to a K-equation.

We fix a base curve X and let $\Lambda f=0$ be a K-equation on X with ω_1, ω_2 a fixed K-basis for Λ. We will associate to the triple $(\Lambda, \omega_1, \omega_2)$ a unique basic elliptic surface over X with functional invariant $\mathcal{J}=J(\omega_1/\omega_2)$ and homological invariant G equal to the monodromy representation for ω_1, ω_2.

We begin with a number of observations and definitions. Suppose $\mathcal{J}\in K(X)$ is any non-constant function on X, and let S be a finite set of points including everywhere $\mathcal{J}=0, 1, \infty$. We denote by X_0 the set $X-S$. The function $\mathcal{J}$ naturally determines a projective representation $\pi_1(X_0)\to PSL_2(\boldsymbol{Z})$ (see Corollary II.2.2). Let G be any $SL_2(\boldsymbol{Z})$ representation of $\pi_1(X_0)$ projectively equivalent to the above (conjugation by elements of $PSL_2(\boldsymbol{Z})$). Because of the preferred choice of sign given by the representation of the SK-equation, $\Lambda_{(\mathcal{J}, 1)}$, we can reduce G to the data included in a map $\pi_1(X_0)\to \boldsymbol{Z}_2$.

Let $\mathcal{F}(\mathcal{J}, G)$ denote the family of all elliptic surfaces (not necessarily algebraic) without exceptional curves of the first kind in the fibres and free of multiple singular fibres, having functional invariant $\mathcal{J}\in K(X)$ (assumed non-constant) and homological invariant G (see Part I, Section 1).

Now given $\mathcal{J}$ and a compatible G, Kodaira (Kodaira [**11**]) constructs a unique basic surface in $\mathcal{F}(\mathcal{J}, G)$. We would like to perform such a construction, given the triple $(\Lambda, \omega_1, \omega_2)$, within our framework of differential equations and show that we obtain all basic surfaces from K-equations in this manner.

Suppose given the triple $(\Lambda, \omega_1, \omega_2)$. Let $X_0\subset X$ be a Zariski open set on which ω_1, ω_2 are holomorphic, non-vanishing. It is easy to construct an analytic family of elliptic curves E_0 over X_0 (see Theorem III.2.6 for details) with a section over X_0, $\mathcal{J}=J(\omega_1/\omega_2)$, and G equal to the monodromy representation associated to ω_1, ω_2.

THEOREM III.1.3. *E_0 over X_0 is algebraic. In fact, there is a unique basic elliptic surface E/X such that the diagram*

$$\begin{array}{ccc} E_0 & \longrightarrow & E \\ \text{section}\ \uparrow\downarrow \pi & & \pi \downarrow\uparrow\ \text{section} \\ X_0 & \longrightarrow & X \end{array}$$

commutes and preserves the section. E/X will be the unique basic surface in the family $\mathfrak{F}(\mathcal{J}, G)$ where $\mathcal{J}$ is $J(\omega_1/\omega_2)$ and G is equal to the monodromy of ω_1, ω_2 over X_0.

PROOF. Suppose that we can find a smooth algebraic compactification of E_0, that is, a complete smooth surface E algebraic over $\boldsymbol{C}$ so that the diagram

$$(1.1)\qquad \text{section}\ \left(\begin{array}{ccc} E_0 & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \pi \\ X_0 & \longrightarrow & X \end{array}\right. \qquad\qquad \pi^{-1}(X_0)=E_0$$

commutes. We can assume E has no exceptional curves of the first kind in the fibres by blowing down. It is clear that the section extends to one from $X \to E$, and that therefore E has no multiple fibres and is the unique basic surface in $\mathfrak{F}(\mathcal{J}, G)$.

In the event we find a complete but singular compactification E as above (1.1), we can apply resolution of singularities and arrive back in the smooth case.

Thus given $(\Lambda, \omega_1, \omega_2)$ it suffices to find some compactification of E_0/X_0. We remark that if $g \in K(X)$ then the differential equation Λ_g satisfied by $g\omega_1$, $g\omega_2$ is also a K-equation and $g\omega_1$, $g\omega_2$ is a K-basis for it with the same $\mathcal{J}$ and same representation. Thus the triple $(\Lambda_g, g\omega_1, g\omega_2)$ yields the same surface as $(\Lambda, \omega_1, \omega_2)$ *a priori* over a possibly smaller set. g is simply a harmless factor of homothety on some open subset of X_0.

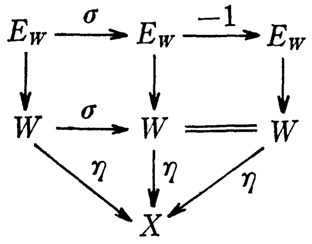
By Theorem II.2.5 there is a λ, $\lambda^2 \in K(X)$, so that Λ is given by $\mathcal{J}$, λ as in that theorem. We indicate this by writing $\Lambda = \Lambda_{(\mathcal{J}, \lambda)}$. Suppose $\lambda \in K(X)$, then by our remark above we can assume without loss of generality that $\lambda = 1$ and we are in the SK-case. Recall the family of elliptic curves over $\boldsymbol{P}^1_{\boldsymbol{C}}$, which we denote E^s, as constructed in Part I, Section 4. We regard $\mathcal{J} = J(\omega_1/\omega_2)$ as a map $X \xrightarrow{\mathcal{J}} \boldsymbol{P}^1_{\boldsymbol{C}}$. We have

$$\begin{array}{ccc} E & \longrightarrow & E^s \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{\ \mathcal{J}\ } & \boldsymbol{P}^1_{\boldsymbol{C}} \end{array}$$

where $E = X \underset{\boldsymbol{P}^1_{\boldsymbol{C}}}{\times} E^s$. E may have singularities, but clearly over a suitable Zariski open set of X both E and E_0 agree. We now resolve singularities and blow down as per our remarks.

If $\lambda \notin K(X)$, then as $\lambda^2 \in K(X)$, λ determines a cover of X, call it W, of degree two. Lifting Λ back to W yields a K-equation on W which now differs from an SK-equation on W by $\lambda \in K(W)$. Thus we find ourselves in the first case considered and we construct a basic elliptic surface E_W over W. Now the involution σ of W over X induces a map $E_W \xrightarrow{\sigma} E_W$ also denoted σ. We compose this map with the natural involution on E_W over W. Thus we have

(1.2)



Let W_0 be a Zariski open subset of W and X_0 be a Zariski open subset of X both chosen small enough so that $W_0 = \pi^{-1}(X_0)$ is étale over X_0, and so that on X_0, ω_1, ω_2 are holomorphic non-vanishing multi-valued. Let $w_0 \in W_0$ and t in the fibre of E_W over w_0. The map $-1\circ\sigma$ sends (w_0, t) to $(\sigma w_0, -t)$ where we identify the fibres over w_0 and σw_0 by σ on E_W and then involute. Thus we see that the quotient of E_W by this automorphism is a possibly singular surface over X which over X_0 is a family of elliptic curves with a section (as $-1\circ\sigma$ preserves the section of E_W), with functional invariant $\mathcal{J}$, and with G equal the monodromy representation of ω_1, ω_2. (The effect of involution is to account for the effect of λ.) We now desingularize and blow down as per our remarks.

Thus to $(\Lambda, \omega_1, \omega_2)$ we associate a unique basic surface. Obviously, if we pass to a $\boldsymbol{Z}$-equivalent basis we obtain the same surface. Further as pointed out in the proof the triple $(\Lambda_g, g\omega_1, g\omega_2)$, $g \in K(X)$ gives the same surface. (See Definition III.1.5 below).

Finally it remains to show that every basic surface arises in this manner.

Fixing $\mathcal{J}$ we have, as mentioned, a natural choice for lifting our projective ($PSL_2(\boldsymbol{Z})$) representation to an $SL_2(\boldsymbol{Z})$ representation by using the SK-equation with solutions $\Phi_1(\mathcal{J})$, $\Phi_2(\mathcal{J})$ (see Chapter II, Section 2). The factor λ allows us to pass to a projectively equivalent representation. We must show that every such representation arises via a λ.

LEMMA III.1.4. *Let $X_0 \subset X$ be any Zariski open subset and let $\pi_1(X_0) \xrightarrow{\rho} \boldsymbol{Z}_2$ be given. Then there exists an algebraic function with λ^2 rational whose monodromy is the representation ρ.*

PROOF. We omit the proof—it is straightforward.

Thus as we run through all pairs $\mathcal{J}$, λ we pick up all pairs $\mathcal{J}$, G. Therefore:

THEOREM III.1.5. *Every basic surface arises from a K-equation.*

PROOF. The pair $\mathcal{J}$, G determines a representation of $\pi_1(X_0) \rightarrow SL_2(\boldsymbol{Z})$ for a suitable Zariski open subset $X_0 \subset X$. This representation must be projectively equivalent to the monodromy representation of the SK-equation $\Lambda_{(\mathcal{J},1)}$ with solutions $\Phi_1(\mathcal{J})$, $\Phi_2(\mathcal{J})$. The difference in signs can be adjusted for using the λ of the lemma. Thus the K-equation $\Lambda=\Lambda_{(\mathcal{J},\lambda)}$ with K-basis $\omega_1=\lambda\Phi_1(\mathcal{J})$, $\omega_2=\lambda\Phi_2(\mathcal{J})$ gives a triple $(\Lambda, \omega_1, \omega_2)$ with $\mathcal{J}=J(\omega_1/\omega_2)$ and monodromy representation G. We can then construct the basic elliptic surface via the method of Theorem III.1.3.

DEFINITION III.1.5. Two triples $(\Lambda, \omega_1, \omega_2)$ and $(\tilde{\Lambda}, \tilde{\omega}_1, \tilde{\omega}_2)$ will be called $\boldsymbol{Z}$-trivially equivalent if there exists a $g \in K(X)$ so that $\tilde{\Lambda}=\Lambda_g$ and $\tilde{\omega}_1$, $\tilde{\omega}_2$ is $\boldsymbol{Z}$-equivalent to $g\omega_1$, $g\omega_2$. This is an equivalence relation.

Theorem III.1.3 and Theorem III.1.5 enable us to construct a surjective map:

$$\text{triples}\,(\Lambda, \omega_1, \omega_2)\,/\,\boldsymbol{Z}\text{-trivial equivalence} \xrightarrow{\chi} \text{basic surfaces.} \tag{1.3}$$

THEOREM III.1.6. *The map χ is injective and is therefore a 1-1 correspondence between the set of basic elliptic surfaces on X (with $\mathcal{J}$ non-constant) and the set of triples $(\Lambda, \omega_1, \omega_2)$ modulo $\boldsymbol{Z}$-trivial equivalence.*

PROOF. Say $(\tilde{\Lambda}, \tilde{\omega}_1, \tilde{\omega}_2)$ and $(\Lambda, \omega_1, \omega_2)$ give the same surface E/X. Then $\mathcal{J}=J(\omega_1/\omega_2)=J(\tilde{\omega}_1/\tilde{\omega}_2)$. So without loss of generality we may assume $\tilde{\omega}=\tilde{\omega}_1/\tilde{\omega}_2=\omega_1/\omega_2=\omega$. This insures the representations for ω_1, ω_2 and $\tilde{\omega}_1$, $\tilde{\omega}_2$ are projectively equal. Therefore, at least locally, there is a function λ such that $\lambda\tilde{\omega}_1=\omega_1$ and $\lambda\tilde{\omega}_2=\omega_2$. Comparing the Wronskians W, $\tilde{W}$ of Λ, $\tilde{\Lambda}$ respectively gives $\lambda^2\tilde{W}=W$. It follows that λ^2 is rational. Alternatively since the monodromy representations of $\tilde{\omega}_1$, $\tilde{\omega}_2$ and ω_1, ω_2 are projectively equal, we must have

$$\pm\begin{pmatrix}\tilde{\lambda} & 0\\ 0 & \tilde{\lambda}\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\tilde{\omega}_1\\ \tilde{\omega}_2\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$$

after analytic continuation—here $\tilde{\lambda}$ is the continuation of λ. Thus

$$\pm\begin{pmatrix}\tilde{\lambda} & 0\\ 0 & \tilde{\lambda}\end{pmatrix}\begin{pmatrix}\tilde{\omega}_1\\ \tilde{\omega}_2\end{pmatrix}=\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}=\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}\tilde{\omega}_1\\ \tilde{\omega}_2\end{pmatrix}.$$

Hence $\tilde{\lambda}=\pm\lambda$, and λ^2 is therefore single-valued. The fact that it is rational then follows from the fact that Λ, $\tilde{\Lambda}$ have regular singular points, so that ω_1, ω_2 and $\tilde{\omega}_1$, $\tilde{\omega}_2$ satisfy appropriate growth estimates near the singular points. We claim $\lambda \in K(X)$. To show this, let ρ, $\tilde{\rho}$ be the representations $\pi_1(X_0, x_0) \to SL_2(\boldsymbol{Z})$ associated to ω_1, ω_2 and $\tilde{\omega}_1$, $\tilde{\omega}_2$ respectively. Since they give isomorphic locally constant $\boldsymbol{Z}\oplus\boldsymbol{Z}$ sheaves on X_0 (namely G), they are conjugate by an element in $\mathrm{Aut}_{\boldsymbol{Z}}(\boldsymbol{Z}\oplus\boldsymbol{Z})=GL_2(\boldsymbol{Z})$. One can show that this forces ρ actually equal to $\tilde{\rho}$ and it follows that $\lambda \in K(X)$. This means $(\Lambda, \omega_1, \omega_2)$ and $\tilde{\Lambda}(, \tilde{\omega}_1, \tilde{\omega}_2)$ are $\boldsymbol{Z}$-trivially equivalent.

§ 2. K-equations and geometry.

In Part II we defined the notion of K-equation by requiring the existence of a special basis of solutions. Of course many such bases exist in general; for example $\boldsymbol{Z}$-equivalent bases. In this section we will analyze all K-bases of a fixed K-equation, and we will see that there may exist non-$\boldsymbol{Z}$-equivalent ones. This will allow us to determine when two elliptic surfaces $/X$ define the same flat vector bundle, or what is the same, when two elliptic surfaces give rise to equivalent representations—equivalence being the usual equivalence of two-dimensional complex representations.

Let X be our base curve and $\Lambda f=0$ a K-equation on X with K-basis ω_1, ω_2. We have $\mathcal{J}=J(\omega_1/\omega_2)\in K(X)$. Suppose that $\Lambda f=0$ has another K-basis of solutions $\tilde{\omega}_1$, $\tilde{\omega}_2$ with $\mathcal{J}=J(\tilde{\omega}_1/\tilde{\omega}_2)\in K(X)$. Fix a Zariski open subset X_0 of X on which both bases are holomorphic nonvanishing and fix a base point $x_0\in X_0$. We know that

$$M\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}=\begin{pmatrix}\tilde{\omega}_1\\ \tilde{\omega}_2\end{pmatrix} \tag{2.1}$$

for some $M\in GL_2(\boldsymbol{C})$, but due to the special conditions imposed on a K-basis, additional conditions are imposed on M.

We first observe:

PROPOSITION III.2.1. *The image of the multivalued map* $\omega=\omega_1/\omega_2: X_0\to\mathfrak{H}$ *is dense in* $\mathfrak{H}$, *the upper half plane.*

PROOF. Clear.

We will feel free to pass from $\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$ or $\begin{pmatrix}\tilde{\omega}_1\\ \tilde{\omega}_2\end{pmatrix}$ to any $\boldsymbol{Z}$-equivalent basis. This is perfectly harmless, changing neither $\mathcal{J}$, $\tilde{\mathcal{J}}$ nor the associated basic surfaces E, $\tilde{E}$ over X (see Part III, Section 1). The effect of such a change will be to allow arbitrary multiplication of M on the left or right by scalars, elements of $SL_2(\boldsymbol{Z})$, or products thereof. So if Ω is the subgroup of $GL_2(\boldsymbol{C})$ generated by scalars and $SL_2(\boldsymbol{Z})$, we will be interested only in the equivalence class of M modulo the double coset action of Ω on $GL_2(\boldsymbol{C})$.

PROPOSITION III.2.2. *M has a representative in* $SL_2(\boldsymbol{R})$.

PROOF. As a linear fractional transformation M must send the values of $\omega=\omega_1/\omega_2$ which have positive imaginary part to the values of $\tilde{\omega}=\tilde{\omega}_1/\tilde{\omega}_2$ which also do. Further, as we approach a logarithmic singularity of Λ in a sector both ω and $\tilde{\omega}$ tend to a "cusp" on $\boldsymbol{R}\cap\{\infty\}$. It is clear then, using Proposition III.2.1 and the fact the monodromy groups have finite index in $SL_2(\boldsymbol{Z})$ (Corollary II.2.3), that $M\in PSL_2(\boldsymbol{R})$. Thus $M\in SL_2(\boldsymbol{R})$ up to scalars.

There is one further condition on M, namely if Γ is the image of $\pi_1(X_0, x_0)$

in $SL_2(\boldsymbol{Z})$ corresponding to the representation for ω_1, ω_2 (the monodromy group), then we must have

$$M\Gamma M^{-1}\subset SL_2(\boldsymbol{Z}). \tag{2.2}$$

This is because $M\Gamma M^{-1}$ should be the monodromy group $\tilde{\Gamma}$ for $\tilde{\omega}_1$, $\tilde{\omega}_2$. Of course this holds if $M\in\Omega$, being nothing more than $\boldsymbol{Z}$-equivalence.

We wish to extract more information about M from condition (2.2). We take $M\in SL_2(\boldsymbol{R})$. Since $\Gamma\subset SL_2(\boldsymbol{Z})$ has finite index, the cosets $\Gamma\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}$, $h\in\boldsymbol{Z}$ cannot all be distinct. Thus $\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\in\Gamma$ for some $b>0$, $b\in\boldsymbol{Z}$. If $M=\begin{pmatrix}w & x\\ y & z\end{pmatrix}$ we have:

$$\begin{pmatrix}w & x\\ y & z\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}z & -x\\ -y & w\end{pmatrix}=\begin{pmatrix}1-byw & bw^2\\ -by^2 & 1+byw\end{pmatrix}\in SL_2(\boldsymbol{Z}).$$

Thus $w=r_w\sqrt{m_w}$ where $r_w\in\boldsymbol{Q}$ and $m_w\in\boldsymbol{Z}$, $m_w>0$ and square free. Likewise $y=r_y\sqrt{m_y}$. If one of w or y is zero we may assume $m_w=m_y$ by taking r_w or $r_y=0$. If $wy\neq 0$, then $\sqrt{m_w}\sqrt{m_y}\in\boldsymbol{Q}$ since $1-byw\in\boldsymbol{Z}$, but both m_w and m_y are positive square free, forcing $m_w=m_y=m$. Now consider the cosets $\begin{pmatrix}1 & 0\\ h & 1\end{pmatrix}$, $h\in\boldsymbol{Z}$. As above we have $z=r_z\sqrt{m'}$ and $x=r_x\sqrt{m'}$, r_z, $r_x\in\boldsymbol{Q}$ and $m'\in\boldsymbol{Z}$ positive square free. Thus

$$M=\begin{bmatrix}r_w\sqrt{m} & r_x\sqrt{m'}\\ r_y\sqrt{m} & r_z\sqrt{m'}\end{bmatrix}$$

but det $M=1$ by assumption, so $(r_wr_z-r_yr_x)\sqrt{m}\sqrt{m'}=1$. Again it follows $m=m'$, hence

$$M=\begin{pmatrix}\sqrt{m} & 0\\ 0 & \sqrt{m}\end{pmatrix}\begin{pmatrix}r_w & r_x\\ r_y & r_z\end{pmatrix} \tag{2.3}$$

where

$$\begin{pmatrix}r_w & r_x\\ r_y & r_z\end{pmatrix}\in GL_2^+(\boldsymbol{Q}).$$

THEOREM III.2.3. *M has a representative in $GL_2^+(\boldsymbol{Q})$, or after scalar adjustment, in $M_n(\boldsymbol{Z})$ with the gcd of the entries being one, that is, a primitive integer matrix of determinant n.*

So after passing to $\boldsymbol{Z}$-equivalent bases

$$M\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}=\begin{pmatrix}\tilde{\omega}_1\\ \tilde{\omega}_2\end{pmatrix} \tag{2.4}$$

with $M\in M_n(\boldsymbol{Z})$. If L_x is the lattice $\{m\omega_1(x)+n\omega_2(x),\ m,\ n\in\boldsymbol{Z}\}\subset\boldsymbol{C}$ and $\tilde{L}_x$ the similar lattice for $\tilde{\omega}_1$, $\tilde{\omega}_2$ then

$$\tilde{L}_x \subset L_x \subset \boldsymbol{C}, \qquad x \in X_0 .$$

This means the fibres E_x and $\tilde{E}_x$ are isogeneous, $\boldsymbol{C}/\tilde{L}_x \to \boldsymbol{C}/L_x$, of degree n. Before investigating this phenomenon more closely we make several comments.

First if $M \in M_n(\boldsymbol{Z})$ but $M \notin SL_2(\boldsymbol{Z})$ then clearly $\mathcal{G} \neq \tilde{\mathcal{G}}$. On the other hand by Theorem II.2.5 there exist functions λ, $\tilde{\lambda}$ with λ^2, $\tilde{\lambda}^2 \in K(X)$ such that $\mathcal{G}$, λ and $\tilde{\mathcal{G}}$, $\tilde{\lambda}$ both express $\Lambda f=0$ using the formula of that result!

PROPOSITION III.2.4. *If $\Gamma = SL_2(\boldsymbol{Z})$ then any two K-bases are $\boldsymbol{Z}$-equivalent and the associated $\mathcal{G}$ (and therefore λ^2) is unique.*

PROOF. The proof is quite similar to that of Theorem III.2.3 using $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. The proposition being equivalent to the statement that the only matrices $M \in SL_2(\boldsymbol{R})$ with $MSL_2(\boldsymbol{Z})M^{-1} \subset SL_2(\boldsymbol{Z})$ are already in $SL_2(\boldsymbol{Z})$.

DEFINITION III.2.5. We will refer to the relation between K-bases discussed above as $\boldsymbol{Q}$-equivalence since the representations into $SL_2(\boldsymbol{Z})$ are conjugate by an element of $GL_2^+(\boldsymbol{Q})$. $\boldsymbol{C}$-equivalence or just equivalence will be the usual equivalence of two-dimensional complex representations.

We now wish to prove:

THEOREM III.2.6. *Let $\Lambda f=0$ be a K-equation on X with two K-bases ω_1, ω_2 and $\tilde{\omega}_1$, $\tilde{\omega}_2$ related by $M\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \end{pmatrix}$ with $M \in M_n(\boldsymbol{Z})$ and primitive. Then there is a rational map ϕ of degree n from the basic elliptic surface $\tilde{E}$ associated to $(\tilde{\Lambda}, \tilde{\omega}_1, \tilde{\omega}_2)$ to the basic elliptic surface E associated to $(\Lambda, \omega_1, \omega_2)$ which over some Zariski open subset $X_0 \subset X$ is a fibre by fibre isogeny. Hence the diagrams*

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\ \phi\ } & E \\ & {\scriptstyle\tilde{\pi}}\searrow \ \swarrow{\scriptstyle\pi} & \\ & X & \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} \tilde{E}_0 & \xrightarrow{\ \phi\ } & E_0 \\ & {\scriptstyle\tilde{\pi}}\searrow \ \swarrow{\scriptstyle\pi} & \\ & X_0 & \end{array} \tag{2.5}$$

where $\tilde{E}_0 = \tilde{\pi}^{-1}(X_0)$, $E_0 = \pi^{-1}(X_0)$ and ϕ is regular on $\tilde{E}_0$. Thus ϕ gives an isogeny of the generic fibres over $K(X)$ of degree n.

PROOF. By passage to $\boldsymbol{Z}$-equivalent bases, we may take $M = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$ (see Lang [13] p. 51). Let $X_0 \subset X$ be a Zariski open subset on which both bases are holomorphic non-vanishing. Let U_0 be the universal cover of X_0 and $\pi_1(X_0)$ the fundamental group acting as a group of covering translations, $\gamma(\tau(u)) = (\gamma\tau)(u)$, $u \in U_0$ and γ, $\tau \in \pi_1(X_0)$. Also $\pi_1(X_0)$ acts on $\omega(u) = \omega_1(u)/\omega_2(u)$ by

$$\omega(u) \longrightarrow M_\gamma \omega(u) = \frac{a_\gamma \omega(u) + b_\gamma}{c_\gamma \omega(u) + d_\gamma}$$

where $\gamma\in\pi_1(X_0)$ and M_γ is the monodromy matrix. Let $f_\gamma(u)=(c_\gamma\omega(u)+d_\gamma)^{-1}$ then $f_{\gamma\beta}(u)=f_\gamma(\beta u)f_\beta(u)$, γ, $\beta\in\pi_1(X_0)$. For every triple (γ, n_1, n_2), $\gamma\in\pi_1(X_0)$, n_1, $n_2\in\boldsymbol{Z}$ define an automorphism $g(\gamma, n_1, n_2)$ of $U_0\times\boldsymbol{C}$ by

$$(u, t)\xrightarrow{g(\gamma, n_1, n_2)}(\gamma u, f_\gamma(u)(t+n_1\omega(u)+n_2)).$$

We have $g(\gamma, n_1, n_2)g(\beta, m_1, m_2)(u, t)=g(\gamma\beta, a_\beta n_1+c_\beta n_2+m_1, b_\beta n_1+d_\beta n_2+m_2)$ where

$$M_\beta=\begin{pmatrix} a_\beta & b_\beta \\ c_\beta & d_\beta \end{pmatrix}$$

is the monodromy matrix around β. The energetic reader can check that this set of automorphisms is actually a group $\mathcal{G}$ with this composition law. Its action is proper discontinuous fixed point free, so that the quotient $U_0\times\boldsymbol{C}/\mathcal{G}$ is a complex manifold which is clearly a family of elliptic curves over X_0 locally given by the lattice $\omega(u)$, 1 or $\omega_1(u)$, $\omega_2(u)$.

Let $\gamma\in\pi_1(X_0)$ then under continuation $u\to\gamma u$, $\omega(u)\to M_\gamma\omega(u)$, $1\to 1$, but the fibre at u is identified with the fibre at γu via the factor of homothety $(c_\gamma\omega(u)+d_\gamma)^{-1}$. Thus $M_\gamma\omega(u)$ identifies with $a_\gamma\omega(u)+b_\gamma$ and 1 identifies with $c_\gamma\omega(u)+d_\gamma$ so that M_γ is the monodromy representation and/or homological invariant about γ of this family of elliptic curves over X_0. It is clear that this family is E_0/X_0 where $E\xrightarrow{\pi}X$ is the basic surface associated to $(\Lambda, \omega_1, \omega_2)$ and $E_0=\pi^{-1}(X_0)$. Similarly we construct $\tilde{E}_0$ over X_0. The above construction is well-known (Kodaira [11]).

We now define ϕ. Consider the identity map $U_0\times\boldsymbol{C}\to U_0\times\boldsymbol{C}$. We will show that this map yields a well-defined analytic map $\phi:\tilde{E}_0\to E_0$. Let $\tilde{g}(\gamma, n_1, n_2)$ be one of the automorphisms used above in constructing $\tilde{E}_0$. We have

$$\begin{array}{ccc} (u, t)\in U_0\times\boldsymbol{C} & \longrightarrow & (u, t)\in U_0\times\boldsymbol{C} \\ \downarrow & & \\ (\gamma u, \tilde{f}_\gamma(u)(t+n_1\tilde{\omega}(u)+n_2))\in U_0\times\boldsymbol{C} & \longrightarrow & (\gamma u, \tilde{f}_\gamma(u)(t+n_1\tilde{\omega}(u)+n_2))\in U_0\times\boldsymbol{C}\,. \end{array}$$

It suffices to find $g(\gamma, m_1, m_2)$ such that

$$g(\gamma, m_1, m_2)(u, t)=(\gamma u, \tilde{f}_\gamma(u)(t+n_1\tilde{\omega}(u)+n_2)).$$

Now let $m_1=n_1n$ and $m_2=n_2$ $\left(M=\begin{pmatrix} n & 0 \\ 0 & 1\end{pmatrix}\right)$. We are reduced to checking:

$$(\gamma u, f_\gamma(u)(t+n_1n\omega(u)+n_2))=(\gamma u, f_\gamma(u)(t+n_1\tilde{\omega}(u)+n_2)).$$

But $\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \end{pmatrix}$ so $n\omega(u) = \tilde{\omega}(u)$. Thus it remains to check $f_\gamma(u) = \tilde{f}_\gamma(u)$. Let

$$M_\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$$

be the monodromy around γ for ω_1, ω_2, then

$$MM_\gamma M^{-1} = \begin{pmatrix} a_\gamma & bn_\gamma \\ \dfrac{1}{n} c_\gamma & d_\gamma \end{pmatrix} = \begin{pmatrix} \tilde{a}_\gamma & \tilde{b}_\gamma \\ \tilde{c}_\gamma & \tilde{d}_\gamma \end{pmatrix}$$

is the monodromy of $\tilde{\omega}_1$, $\tilde{\omega}_2$. Finally, we see that $f_\gamma(u) = (c_\gamma \omega(u) + d_\gamma)^{-1} = ((c_\gamma/n)n\omega(u) + d_\gamma)^{-1} = \tilde{f}_\gamma(u)$. Hence we have an analytic map

$$\begin{array}{ccc} \tilde{E}_0 & \xrightarrow{\ \phi\ } & E_0 \\ & {\scriptstyle\tilde{\pi}}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & X_0 & \end{array}$$

which is an isogeny of degree n fibre by fibre.

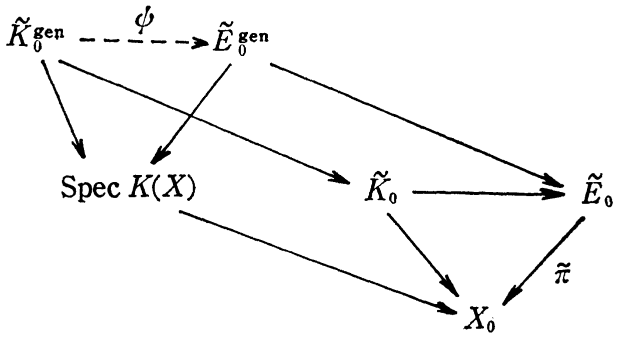
Let $\tilde{K}_0 \subset \tilde{E}_0$ be the "kernel" of ϕ, that is, if s denotes the section $s : X_0 \to E_0$, $\tilde{K}_0 = \phi^{-1}(s(X_0))$. It consists of exactly n points in each fibre of $\tilde{E}_0$ over X_0. Now $\tilde{K}_0$ is a closed algebraic subvariety of $\tilde{E}_0$ because $\tilde{K}_0 = \tilde{E}_0 \cap \tilde{K}$ where $\tilde{K}$ is the union of the closure of $\tilde{K}_0$ in $\tilde{E}$ and all the fibres of $\tilde{E}$ over the points in $X - X_0$. $\tilde{K}$ is obviously closed in $\tilde{E}$ which is a complete smooth surface (hence projective) and so by Chow's Theorem $\tilde{K}$ is algebraic. Also $\tilde{E}_0 \subset \tilde{E}$ is Zariski open, so that $\tilde{K}_0 = \tilde{E}_0 \cap \tilde{K}$ is algebraic.

Clearly $\tilde{K}_0$ is an étale cover of X_0 (not necessarily connected). The generic fibre $\tilde{K}_0^{\mathrm{gen}}$,

$$\begin{array}{ccc} \tilde{K}_0^{\mathrm{gen}} & \longrightarrow & \tilde{K}_0 \\ \text{étale}\downarrow & & \downarrow\text{étale} \\ \operatorname{Spec} K(X) & \longrightarrow & X_0 \end{array}$$

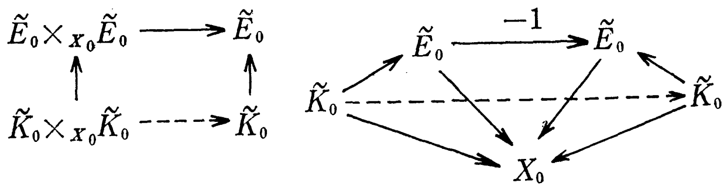
is therefore isomorphic to $\operatorname{Spec}(\bigoplus_{i=1}^{r} K_i)$ where each K_i is a finite separable field extension of $K(X)$ with $\sum_{i=1}^{r} [K_i : K(X)] = n$. We also have the diagram

(2.6)



where $\tilde{E}_0^{\text{gen}}$ is the generic fibre of $\tilde{E}_0 \xrightarrow{\tilde{\pi}} X_0$ which is an elliptic curve over $K(X)$, the section furnish a $K(X)$-rational point. (It is of course just $\tilde{E}_0^{\text{gen}}$.) By the universal property of fibre products in the category of schemes the diagram (2.6) can be completed with a map ϕ so that the top square is a fibre product. Thus $\tilde{K}_0$ cuts out a divisor on the generic curve $\tilde{E}_0^{\text{gen}}$ over $K(X)$, geometrically (that is, over an algebraic closure of $K(X)$) consisting of n points each taken with multiplicity one. Moreover ϕ is a closed immersion over Spec $K(X)$, so it exhibits $\tilde{K}_0^{\text{gen}}$ as a closed subscheme (reduced) of $\tilde{E}_0^{\text{gen}}$ rational over $K(X)$. In addition, $\tilde{K}_0$ can be regarded as a group scheme over X_0. This is because $\tilde{E}_0$ is a group scheme over X_0 and

$\tilde{s}: X_0 \longrightarrow \tilde{K}_0 \subset \tilde{E}_0$



the section, multiplication, and inverse all descend to $\tilde{K}_0$ since they do analytically.

We finally conclude that this divisor on $\tilde{E}_0^{\text{gen}}$ which we identify with $\tilde{K}_0^{\text{gen}}$ is a closed algebraic subgroup of $\tilde{E}_0^{\text{gen}}$ rational over $K(X)$. We now apply the following result (Weil [**24**]): If G is an algebraic group defined over a field K and H is a closed normal subgroup also defined over K then G/H can be given the structure of an algebraic group over K and the quotient map $G \xrightarrow{\phi} G/H$ will be defined over K. It now follows that our map $\tilde{E}_0 \xrightarrow{\phi} E_0$ over X_0 is algebraic.

To elaborate further on this result, let G, $\tilde{G}$ be the homological invariants of E, $\tilde{E}$ respectively. They are locally constant $\boldsymbol{Z} \oplus \boldsymbol{Z}$ sheaves over X_0. Let $\mathfrak{f}$, $\tilde{\mathfrak{f}}$ be the normal bundles of the sections $s(X)$, $\tilde{s}(X)$ in E, $\tilde{E}$ viewed as line bundles on X restricted to X_0 (Kodaira [**12**]). We have the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \tilde{G} & \longrightarrow & \tilde{\mathfrak{f}} \\ & & \text{“}M\text{”}\downarrow & & \downarrow\cong \\ 0 & \longrightarrow & G & \longrightarrow & \mathfrak{f} \end{array} \qquad M\in M_n(\boldsymbol{Z}) \ \text{ induces “}M\text{”}$$

of sheaves of abelian groups over X_0. Note that G, $\tilde{G}$ when tensored with $\boldsymbol{C}$ (actually $\boldsymbol{Q}$) become isomorphic. This is because over $\boldsymbol{C}$ they are the sheaf of flat sections of Λ. The isomorphism $\tilde{\mathfrak{f}}\cong\mathfrak{f}$ being induced by ϕ which is an etale morphism of $\tilde{E}_0$ to E_0 mapping section to section. Taking kernels and cokernels yields

$$\begin{array}{ccccccccc} & & 0 & & 0 & & A & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{G} & \longrightarrow & \tilde{\mathfrak{f}} & \longrightarrow & \tilde{E}_0 & \longrightarrow & 0 \\ & & \text{“}M\text{”}\downarrow & & \downarrow\cong & & \downarrow\phi & & \\ 0 & \longrightarrow & G & \longrightarrow & \mathfrak{f} & \longrightarrow & E_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & A & & 0 & & 0 & & \end{array}$$

where A is a locally constant $\boldsymbol{Z}/n\boldsymbol{Z}$ sheaf over X_0. This diagram can actually be used to construct ϕ. The map $\tilde{G}\xrightarrow{\text{“}M\text{”}}G$ induces $\tilde{G}\otimes_{\boldsymbol{Z}}\boldsymbol{R}\to G\otimes_{\boldsymbol{Z}}\boldsymbol{R}$ which is an isomorphism $\tilde{\mathfrak{f}}\to\mathfrak{f}$ as rank two real analytic bundles, however locally it is $\boldsymbol{C}$-analytic (ϕ exists locally). Taking cokernels as families of Lie groups yields ϕ.

Without going into detail, we will show how to find an example of a K-equation with two K-bases which are *not* $\boldsymbol{Z}$-equivalent. Consider the principal congruence subgroup

$$\Gamma(2)=\left\{M\in SL_2(\boldsymbol{Z}),\ M\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\bmod 2\right\}$$

and the group

$$\Gamma_0(4)=\left\{M\in SL_2(\boldsymbol{Z}),\ M=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\ c\equiv 0\bmod 4\right\}.$$

Let $M=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$ then

$$\begin{pmatrix}1/2 & 0\\ 0 & 1\end{pmatrix}\Gamma(2)\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}=\Gamma_0(4).$$

It will then be enough to produce a K-equation with some K-basis having monodromy group $\Gamma(2)$. Such equations can be found. (Stiller [**22**]).

REMARK. In general the rational map ϕ of Theorem III.2.6 does not extend to a regular map on all of $\tilde{E}$. Also, this result gives a great deal of information about the torsion points on the generic elliptic curve. For example if the monodromy is $SL_2(\boldsymbol{Z})$, then using Proposition III.2.4, we see that there are no such torsion points.

DEFINITION III.2.7. Two K-equations Λ and $\tilde{\Lambda}$ will be called trivially-equivalent if there exists $g \in K(X)$ such that the solutions of $\tilde{\Lambda}$ are g times those of Λ.

Recall the map χ of Chapter III, Section 1. We have

$$\begin{array}{ccc}
\dfrac{\text{triples } (\Lambda, \omega_1, \omega_2)}{\boldsymbol{Z}\text{-trivial equivalence}} & \xrightarrow[\text{1-1}]{\chi} & \text{basic surfaces} \\
\text{surjective} \downarrow & & \downarrow \text{surjective} \\
\dfrac{K\text{-equations } \Lambda}{\text{trivial equivalence}} & \overset{\tilde{\chi}}{\dashrightarrow} & \dfrac{\text{basic surfaces}}{\text{generic isogeny}}
\end{array} .$$

The results of this section yield the map $\tilde{\chi}$ which is surjective.

§ 3. More on K-equations and geometry.

In the previous section we defined a map $\tilde{\chi}$ from the set of K-equations,

$$\Lambda f = \frac{d^2 f}{dx^2} + P\frac{df}{dx} + Qf = 0,$$

modulo trivial equivalence to the set of basic elliptic surfaces over X modulo generic isogeny. We have shown that $\tilde{\chi}$ is surjective and will show it is injective!

Suppose Λ, $\tilde{\Lambda}$ are two K-equations which give generically isogeneous basic elliptic surfaces. This makes sense since possible choices of K-bases for Λ, $\tilde{\Lambda}$ lead only to generically isogeneous surfaces. We therefore choose K-bases and get basic surfaces E, $\tilde{E}$ and a rational map

$$\begin{array}{ccc}
\tilde{E} & \xrightarrow{\phi} & E \\
& {}_{\tilde{\pi}}\searrow \quad \swarrow_{\pi} & \\
& X &
\end{array}
\qquad\qquad
\begin{array}{ccc}
\tilde{E}_0 = \tilde{\pi}^{-1}(X_0) & \xrightarrow{\phi} & E_0 = \pi^{-1}(X_0) \\
& {}_{\tilde{\pi}}\searrow \quad \swarrow_{\pi} & \\
& X_0 &
\end{array}$$

which is a regular fibre by fibre isogeny over some Zariski open subset X_0 of X. Now ϕ induces an injection $H_1(\tilde{E}_x, \boldsymbol{Z}) \to H_1(E_x, \boldsymbol{Z})$ of the homology of the fibres $\tilde{E}_x$, E_x over $x \in X_0$. The result is an inclusion $\tilde{G} \to G$ of homological invariants over X_0 as sheaves of abelian groups. After tensoring with $\boldsymbol{Q}$ (or $\boldsymbol{C}$) the representations become equal. Thus:

THEOREM III.3.1. *Two K-equations which give generically isogeneous surfaces have the same monodromy* /$\boldsymbol{C}$ (*or* $\boldsymbol{Q}$) *and give the same flat vector bundle over some Zariski open subset of X.*

Now suppose we are given two K-equations Λ, $\tilde{\Lambda}$ with the same monodromy over $\boldsymbol{C}$, that is, over some Zariski open subset X_0 of X both Λ and $\tilde{\Lambda}$ determine the same flat vector bundle. Since these are K-equations we get representations

$$\begin{aligned} \rho &: \pi_1(X_0, x_0) \longrightarrow SL_2(\boldsymbol{Z}) \\ \tilde{\rho} &: \pi_1(X_0, x_0) \longrightarrow SL_2(\boldsymbol{Z}) \end{aligned} \tag{3.1}$$

$x_0 \in X_0$ a base point, with global groups Γ, $\tilde{\Gamma} \subset SL_2(\boldsymbol{Z})$ respectively. Because these representations are equivalent there is an $M \in GL_2(\boldsymbol{C})$ such that $M\rho(\gamma)M^{-1} = \tilde{\rho}(\gamma)$ for every $\gamma \in \pi_1(X_0, x_0)$. We can take M in $SL_2(\boldsymbol{C})$, and we must have $M\Gamma M^{-1} \subset SL_2(\boldsymbol{Z})$.

LEMMA III.3.2. *We can find* $M \in GL_2^+(\boldsymbol{Q})$ *which conjugates* ρ *into* $\tilde{\rho}$.

PROOF. The first part of this argument is virtually identical to that of Theorem III.2.3 where we assumed $M \in SL_2(\boldsymbol{R})$. Allowing $M \in SL_2(\boldsymbol{C})$ leads to

$$M = \begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{m} \end{pmatrix} \begin{pmatrix} r_w & r_x \\ r_y & r_z \end{pmatrix}$$

with m a square free integer positive *or negative* and

$$\begin{pmatrix} r_w & r_x \\ r_y & r_z \end{pmatrix} \in GL_2(\boldsymbol{Q})$$

with determinant $1/m$. We will show that the case $m<0$ really cannot occur for the representations we have considered. Some caution is in order: given a K-basis ω_1, ω_2 we obtain a specific map $\rho : \pi_1(X_0, x_0) \to SL_2(\boldsymbol{Z})$ once a base point $x_0 \in X_0$ and branches ω_1, ω_2 are selected. Now obviously conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ yields another map $\eta : \pi_1(X_0, x_0) \to SL_2(\boldsymbol{Z})$. Over $\boldsymbol{C}$, ρ and η are viewed as equivalent. But clearly η does not arise from a K-basis since $\operatorname{Im}(\omega_2/\omega_1) < 0$ not >0. Since in our situation both ρ and $\tilde{\rho}$ arise from K-bases, this positivity will save us. Namely, without loss of generality, we can assume that around some point $\tilde{\rho}$ gives $\pm\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b>0$, see Proposition II.1.2. (Note $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ occurs for fibre type I_b and $\begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}$ occurs for fibre type I_b^*, see Kodaira [11]). The main point here is that b is positive, being the order of the pole of $\tilde{\mathcal{J}}$. If $m<0$ then let

$$N = \begin{pmatrix} r_w & r_x \\ r_y & r_z \end{pmatrix} \in GL_2^-(\boldsymbol{Q}) .$$

We have

$$N\rho(\gamma)N^{-1}=\tilde{\rho}(\gamma).$$

Let $N^*=\begin{pmatrix}0&1\\1&0\end{pmatrix}N\in GL_2^+(\boldsymbol{Q})$. We now pass to a $\boldsymbol{Q}$-equivalent basis for ρ, that is, if ω_1, ω_2 is the K-basis of Λ giving ρ take the basis $N^*\begin{pmatrix}\omega_1\\\omega_2\end{pmatrix}$, which is another K-basis of Λ, corresponding to a generically isogeneous surface, by the results of Section 2 above. Note we have $SL_2(\boldsymbol{Z})$ monodromy as

$$\rho^{\text{new}}(\gamma)=N^*\rho(\gamma)N^{*-1}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\tilde{\rho}\begin{pmatrix}0&1\\1&0\end{pmatrix}\in SL_2(\boldsymbol{Z}).$$

Our assumption on $\tilde{\rho}$ means that around some point we have:

$$\rho^{\text{new}}(\gamma)=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\pm1&\pm b\\0&\pm1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}=\pm\begin{pmatrix}1&0\\b&1\end{pmatrix},\quad b>0.$$

We conjugate by $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ to see what sort of translation this is

$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\pm1&0\\\pm b&\pm1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\pm\begin{pmatrix}1&-b\\0&1\end{pmatrix}.$$

However ρ^{new} is given by a K-basis. This contradicts the proof of Proposition II.1.2 where it is shown that at a parabolic point the local matrix is a positive translation.

We again take two K-equations Λ, $\tilde{\Lambda}$ with K-bases ω_1, ω_2 and $\tilde{\omega}_1$, $\tilde{\omega}_2$ determining basic surfaces E, $\tilde{E}$ and maps ρ, $\tilde{\rho}:\pi_1(X_0, x_0)\to SL_2(\boldsymbol{Z})$ which we assume yield equivalent representations over $\boldsymbol{C}$ and hence over $\boldsymbol{Q}$ (by an element of $GL_2^+(\boldsymbol{Q})$) as the lemma shows.

We wish to prove:

THEOREM III.3.3. *Two K-equations Λ, $\tilde{\Lambda}$ have the same monodromy over $\boldsymbol{C}$ (or $\boldsymbol{Q}$) if and only if they correspond to generically isogeneous basic surfaces.*

PROOF. If the equations correspond to generically isogeneous basic surfaces, then by Theorem III.3.1 they have the same monodromy. Suppose conversely that they have the same monodromy. Without loss of generality, using Lemma III.3.2, we can choose K-bases of Λ, $\tilde{\Lambda}$, call them ω_1, ω_2 and $\tilde{\omega}_1$, $\tilde{\omega}_2$ respectively, so that the maps ρ, $\tilde{\rho}:\pi_1(X_0, x_0)\to SL_2(Z)$ are *equal*. Lemma III.3.4 below then shows that $\omega_1/\omega_2=\tilde{\omega}_1/\tilde{\omega}_2$, hence $\mathcal{J}=J\circ\omega_1/\omega_2$ and $\tilde{\mathcal{J}}=J\circ\tilde{\omega}_1/\tilde{\omega}_2$ are equal. Now if λ were not in $K(X)$ then for some path $\gamma\in\pi_1(X_0, x_0)$ we would have λ continuing to $-\lambda$. This would force $\rho(\gamma)=-\tilde{\rho}(\gamma)$ a contradiction. This means that $\lambda\in K(X)$ and the results of Part III, Section 1 show that the surfaces are isomorphic over X.

LEMMA III.3.4. *Let X be any complete smooth curve $/\boldsymbol{C}$ and suppose we are given two multivalued holomorphic non-constant functions $\omega, \tilde{\omega}: X_0 \to \mathfrak{H}$ where $X_0 \subset X$ is some Zariski open set and $\mathfrak{H}$ is the upper half plane. We assume that a base point $x_0 \in X_0$ and branches of $\omega, \tilde{\omega}$ have been chosen so that analytic continuation leads to maps $\rho, \tilde{\rho}: \pi_1(X_0, x_0) \to PSL_2(\boldsymbol{Z})$ which are equal. Then $\omega = \tilde{\omega}$.*

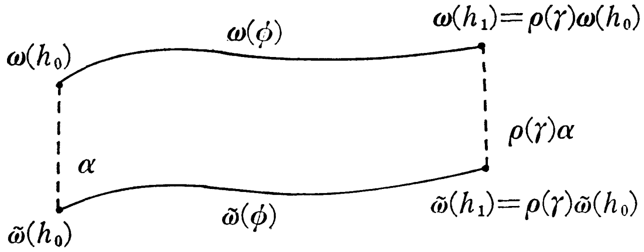
PROOF. Clearly $\omega, \tilde{\omega}$ are the quotient of solutions (K-bases) of some K-equations $\Lambda, \tilde{\Lambda}$. We assume first that the monodromy group Γ, which is the image of $\pi_1(X_0, x_0)$ in $PSL(\boldsymbol{Z})$, has no elliptic elements. Thus $\omega, \tilde{\omega}$ have only parabolic local monodromy. Since they have the same monodromy we can choose X_0 to be X minus these points with parabolic monodromy. Let $\Pi \subset PSL_2(\boldsymbol{R})$ serve to uniformize X_0.

$$\left(\text{Note } \ \#(X - X_0) \geqq \begin{cases} 3 & \text{if } X = \boldsymbol{P}^1_{\boldsymbol{C}} \\ 1 & \text{otherwise.} \end{cases}\right)$$

We have

$$\begin{array}{ccc} \mathfrak{h} & \overset{\omega}{\underset{\tilde{\omega}}{\rightrightarrows}} & \mathfrak{h} \\ \downarrow & & \downarrow \\ \mathfrak{h}/\Pi \cong X_0 & \overset{\sigma}{\underset{\tilde{\sigma}}{\rightrightarrows}} & \mathfrak{h}/\Gamma = C_0 \end{array}$$

where the vertical maps are covering maps and $\sigma, \tilde{\sigma}$ are maps from X_0 to the modular curve $\boldsymbol{C} = \mathfrak{H}/\Gamma$. Note that $\sigma, \tilde{\sigma}$ extend to maps from X to C the compactification of C_0. The missing points $X - X_0$ are precisely the points mapping to the cusps of C under either σ or $\tilde{\sigma}$. Thus $\sigma, \tilde{\sigma}$ are proper surjective and it is easy to see that $\omega, \tilde{\omega}$ are also surjective as maps $\mathfrak{H} \to \mathfrak{H}$. Now let $\gamma \in \pi_1(X_0, x_0)$ lifting to a path ϕ from h_0 to h_1 on $\mathfrak{H}$ the universal cover. Consider the paths $\omega(\phi)$ and $\tilde{\omega}(\phi)$



from $\omega(h_0)$ to $p(\gamma)\omega(h_0)$ and $\tilde{\omega}(h_0)$ to $\rho(\gamma)\tilde{\omega}(h_0)$ (as $\rho = \tilde{\rho}$) respectively. Let α be a path from $\omega(h_0)$ to $\tilde{\omega}(h_0)$ and $\rho(\gamma)\alpha$ its translate by $\rho(\gamma)$ from $\rho(\gamma)\omega(h_0)$ to $\rho(\gamma)\tilde{\omega}(h_0)$. Using α projected to C_0 we have

$$\begin{array}{ccc} & \overset{\sigma_*}{\nearrow} & \pi_1(C_0, c_0) \\ \pi_1(X_0, x_0) & & \updownarrow \alpha \\ & \underset{\tilde{\sigma}_*}{\searrow} & \pi_1(C_0, \tilde{c}_0) \end{array}$$

where $c_0=\sigma(x_0)$ and $\tilde{c}_0=\tilde{\sigma}(x_0)$. It follows that

$$\begin{array}{ccc} & \overset{\sigma_*}{\nearrow} & \pi_1(C, c_0) \\ \pi_1(X, x_0) & & \updownarrow \alpha \\ & \underset{\tilde{\sigma}_*}{\searrow} & \pi_1(C, \tilde{c}_0) \end{array}$$

commutes and that $\sigma, \tilde{\sigma} : X \to C$ are homotopic. They thus induce the same map on homology

$$H_1(X, \boldsymbol{Z}) \xrightarrow{\sigma_*,\ \tilde{\sigma}_*} H_1(C, \boldsymbol{Z}).$$

Let Ω be any differential of the first kind on C, then $\sigma^*(\Omega)=\tilde{\sigma}^*(\Omega)$ since for any class $\gamma \in H_1(X, \boldsymbol{Z})$

$$\int_\gamma \sigma^*(\Omega)=\int_{\sigma_*(\gamma)} \Omega=\int_{\tilde{\sigma}_*(\gamma)} \Omega=\int_\gamma \tilde{\sigma}^*(\Omega).$$

We consider four cases:

Case 1. $g \geqq 2$ and C not hyperelliptic then using the canonical embedding of C shows that $\sigma=\tilde{\sigma}$.

Case 2. $g \geqq 2$, and C hyperelliptic then using the canonical embedding shows $\sigma, \tilde{\sigma}$ differ at most by the involution of C which is not possible since they agree on homology and/or pull-backs of regular 1-forms.

Case 3. $g=1$ then using the universal mapping property of Jacobians and the fact that $\mathrm{Jac}(C) \cong C$, we see that $\sigma, \tilde{\sigma}$ differ by an automorphism of C. However that map must fix the cusps of which there is at least one. Thus the automorphism is a complex multiplication as opposed to a translation. However this forces $\sigma=\tilde{\sigma}$ since any complex multiplication other than the identity gives a non-trivial map on homology.

Case 4. $g=0$ then $\sigma, \tilde{\sigma}$ are rational functions on X. Normalizing three of the cusps to $0, 1, \infty$ we see that $\sigma, \tilde{\sigma}$ have the same zeros, poles, and ones and therefore are equal.

To complete the proof we must allow elliptic elements in Γ. However look at $\overline{\Gamma(2)}\cap\Gamma$ which determines some finite cover of X via ρ^{-1}. Lifting to this cover puts us in the case of no elliptic elements, and the result easily follows.

One can also prove the theorem by showing that the equivalent representations lead to equivalent l-adic representations of the generic fibres and then applying the global isogeny theorem of Deligne, Lang, and Serre to get generically isogeneous surfaces. We can now give this isogeny theorem as a corollary.

COROLLARY III.3.5 (See Lang [**13**]). *Let K be a function field in one variable over the complex numbers. Let E, $\tilde{E}$ be elliptic curves over K, with invariants j, $\tilde{j}$ transcendental over $\boldsymbol{C}$. Assume $V_l(E)$ and $V_l(\tilde{E})$, the extended Tate-modules, are* $\mathrm{Gal}(\bar{K}/K)$ *isomorphic ($\bar{K}$ an algebraic closure of K). Then E, $\tilde{E}$ are isogeneous over K.*

PROOF. One can easily show that if the l-adic representations are the same then the usual representations associated to the basic surfaces given by E, $\tilde{E}$ are the same.

COROLLARY III.3.6. *If two basic surfaces have the same homological invariant $/\boldsymbol{Z}$ then they are isomorphic. That is the homological invariant $/\boldsymbol{Z}$, as opposed to $/\boldsymbol{C}$ or $\boldsymbol{Q}$, actually determines the functional invariant.*

Now we would like to prove:

THEOREM III.3.7. *$\tilde{\chi}$ is injective, that is, two K-equations Λ, $\tilde{\Lambda}$ which yield generically isogeneous basic elliptic surfaces are trivially equivalent.*

PROOF. Fibre by fibre over some Zariski open set X_0 of X we have isogenies. Let $x_0\in X_0$ and near x_0 choose K-bases ω_1, ω_2 and $\tilde{\omega}_1$, $\tilde{\omega}_2$. Because of the isogeny there exists a function $\lambda(x)$ locally such that

$$\lambda(x)(\boldsymbol{Z}\tilde{\omega}_1(x)+\boldsymbol{Z}\tilde{\omega}_2(x))\subset\boldsymbol{Z}\omega_1(x)+\boldsymbol{Z}\omega_2(x)\,.$$

Thus locally

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}=\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \end{pmatrix},$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in M^+(\boldsymbol{Z})$. The usual calculation gives λ^2 rational (see page 22 the proof of Theorem III.1.6.). The only additional point one needs to observe is that $a\omega_1+b\omega_2$, $c\omega_1+d\omega_2$ also form a basis for the solutions of Λ so the Wronskian can be computed from them. Without loss of generality we can assume $\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ by a change of K-basis. Now projectively our representations are equal. If $\lambda\notin K(X)$, then around some path ω_1, ω_2 and $\tilde{\omega}_1$, $\tilde{\omega}_2$ give matrices differing by a sign. However as we have proved, isogeneous surfaces have equivalent representations over $\boldsymbol{Q}$ by an element in $GL_2^+(\boldsymbol{Q})$. One can easily see this yields a contradiction. Thus $\lambda\in K(X)$ and the K-equations are trivially equivalent.

Finally we state:

THEOREM III.3.8. *Two K-equations have the same monodromy if and only if they are trivially equivalent (if and only if they correspond to generically isogeneous basic elliptic surfaces).*

PROOF. Follows from Theorem III.3.3 and Theorem III.3.7.

Thus we know when two K-equations yield the same flat bundle on some $X_0 \subset X$.

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